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A simple proof of the Craig–Sakamoto theorem

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Abstract

We give a simple proof of the Craig–Sakamoto theorem, which asserts that two real symmetric matrices A and B satisfy $\det(I - aA - bB) = \det(I - aA) \det(I - bB)$ for all real numbers a and b if and only if $AB = 0$. © 2000 Elsevier Science Inc. All rights reserved.

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The Craig–Sakamoto theorem on the independence of two quadratic forms can be stated as follows.

Theorem 1. *Two $n \times n$ real symmetric matrices A and B satisfy*

$$\det(I - aA - bB) = \det(I - aA) \det(I - bB) \quad \forall a, b \in \mathbb{R} \quad (1)$$

if and only if $AB = 0$.

One may see [1,3] for the history and the importance of this result, and see [2,4–6] for several proofs of it. The purpose of this note is to give a simple proof of Theorem 1. Our proof depends only on the following well known fact.

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Lemma 2. Suppose $C = (c_{ij})$ is an $n \times n$ real symmetric matrix with the largest eigenvalue equal to λ_1 . Then $c_{ii} \leq \lambda_1$ for all $i = 1, \dots, n$. If $c_{ii} = \lambda_1$, then $c_{ij} = 0 = c_{ji}$ for all $j \neq i$.

For the sake of completeness, we give a short proof of Lemma 2.

Proof of Lemma 2. Suppose C satisfies the hypothesis of the lemma and the largest eigenvalue of C has multiplicity m with $1 \leq m \leq n$. Then there is an orthonormal basis $\{v_1, \dots, v_n\}$ for \mathbb{R}^n such that $Cv_j = \lambda_j v_j$ with $\lambda_1 = \dots = \lambda_m > \lambda_{m+1} \geq \dots \geq \lambda_n$. Let $\{e_1, \dots, e_n\}$ be the standard basis for \mathbb{R}^n . For any i with $1 \leq i \leq n$, there exist $t_1, \dots, t_n \in \mathbb{R}$ with $\sum_{j=1}^n t_j^2 = 1$ such that $e_i = \sum_{j=1}^n t_j v_j$ and $c_{ii} = e_i^T C e_i = \sum_{j=1}^n t_j^2 \lambda_j \leq \lambda_1$. The equality holds if and only if $t_{m+1} = \dots = t_n = 0$, i.e., e_i is an eigenvector of C corresponding to the largest eigenvalue. Thus, $Ce_i = \lambda_1 e_i$, and hence $c_{ii} = \lambda_1$ is the only nonzero entry in the i th column. Since C is symmetric, c_{ii} is also the only nonzero entry in the i th row. \square

We are now ready to present our proof of Theorem 1.

Proof of Theorem 1. The (\Leftarrow) part is clear. We prove the converse by induction on n . The result is clear if $n = 1$. Suppose $n > 1$ and the result is true for symmetric matrices of sizes smaller than n . Let A and B be nonzero $n \times n$ real symmetric matrices satisfying (1). Denote by $\rho(C)$ the spectral radius of a square matrix C . Replacing A by $\pm A/\rho(A)$ and B by $B/\rho(B)$, we may assume that $1 = \rho(A) = \rho(B)$ is the largest eigenvalue of A . Let Q be an orthogonal matrix such that $QAQ^T = I_m \oplus \text{diag}(a_{m+1}, \dots, a_n)$ with $1 > a_{m+1} \geq \dots \geq a_n$. We shall show that $(QAQ^T)(QBQ^T) = 0$ and hence $AB = 0$.

For simplicity, we assume that $Q = I$. Let $b = \pm 1$. If $r > 1$, then both A/r and bB/r have eigenvalues in the open interval $(-1, 1)$. Thus, $I - A/r$ and $I - bB/r$ are invertible, and

$$\det(I - A/r - bB/r) = \det(I - A/r) \det(I - bB/r) \neq 0.$$

Moreover, since

$$\det(I - A - bB) = \det(I - A) \det(I - bB) = 0,$$

we see that 1 is the largest eigenvalue of the matrix $A + bB$ for $b = \pm 1$.

Next, we show that B is of the form $0_m \oplus B_2$. Note that all the first m diagonal entries of A are equal to the largest eigenvalue of $A \pm B$. If the first m diagonal entries of B are not all 0, then the matrix $A + B$ or $A - B$ will have a diagonal entry larger than 1, contradicting Lemma 2. So, all the first m diagonal entries of the matrix $A + B$ equal the largest eigenvalue. By Lemma 2 again, $A + B$ must be of the form $I_m \oplus C_2$. Hence, B is of the form $0_m \oplus B_2$ as asserted.

Now, let $A = I_m \oplus A_2$. Then for any real numbers a and b with $a \neq 1$, we have

$$\begin{aligned} & \det(I_{n-m} - aA_2 - bB_2) \\ &= \det(I_n - aA - bB) / \det(I_m - aI_m) \\ &= \det(I_n - aA) \det(I_n - bB) / \det(I_m - aI_m) \\ &= \det(I_{n-m} - aA_2) \det(I_{n-m} - bB_2). \end{aligned}$$

By continuity, we can remove the restriction that $a \neq 1$. Using the induction assumption, we see that $A_2B_2 = 0$. Hence, we have $AB = 0$ as desired. \square

References

- [1] M.F. Driscoll, W.R. Gundberg Jr., A history of the development of Craig's theorem, *Amer. Statist.* 40 (1986) 65–70.
- [2] M.F. Driscoll, B. Krasnicka, An accessible proof of Craig's theorem in the general case, *Amer. Statist.* 49 (1995) 59–62.
- [3] J. Ogawa, A history of the development of Craig–Sakamoto's theorem from Japanese standpoint, *Proc. Ann. Inst. Statist. Math.* 41 (1993) 47–59.
- [4] I. Olkin, A determinantal proof of Craig–Sakamoto theorem, *Linear Algebra Appl.* 264 (1997) 217–223.
- [5] J.G. Reid, M.F. Driscoll, An accessible proof of Craig's theorem in the noncentral case, *Amer. Statist.* 42 (1988) 139–142.
- [6] O. Taussky, On a matrix theorem of A.T. Craig and H. Hotelling, *Indagationes Mathematicae* 20 (1958) 139–141.